Electromagnetism in a moving chiral medium

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We present manifestly covariant equations and boundary equations for electromagnetism in a moving chiral medium. Then, to solve these equations, we develop a particular formalism based on a four-dimensional extension of the Hertz vector potential satisfying a generalized wave equation. Finally, this formalism is applied to the propagation of plane waves in a homogeneous isotropic medium and to their excitation by a plane wave incident on the boundary of such a medium.

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I. INTRODUCTION

Using Post's constitutive equations [1] we present a covariant formalism of electromagnetism in moving chiral media. We give for a homogeneous isotropic medium the general formulas in terms of the usual electromagnetic fields. Then a particular formalism to solve these covariant equations is introduced. Finally, we discuss the propagation and the excitation of plane waves in a moving homogeneous chiral medium.

We first give some relations that will be useful later. The greek and latin indices take the values 0,1,2,3 and 1,2,3, respectively, and we use the summation convention on repeated indices; $\delta_{\mu\nu}$ is the Kronecker symbol and $\epsilon_{\mu\nu\alpha\beta}$ is the permutation tensor. The components of the metric tensor are

$$g_{00}=1,\quad g_{11}=g_{22}=g_{33}=-1,\quad g_{\mu\nu}=0 \quad \text{for } \mu\neq\nu \ . \tag{1}$$

One has the relations [2]

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\rho\sigma}=2\delta^{\rho\sigma}_{\mu\nu}\ , \tag{2}$$

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\rho\sigma\tau}=\delta^{\rho\sigma\tau}_{\beta\mu\nu}\ , \tag{3}$$

where $\delta^{\rho\sigma}_{\mu\nu}$ and $\delta^{\rho\sigma\tau}_{\beta\mu\nu}$ are 2×2 and 3×3 permutation matrices, respectively.

Now let $G_{\alpha\beta}$ be an antisymmetric tensor and u_μ be a unit 4-vector ($u_\mu u^\mu=1$). Then the following identity is easy to prove:

$$G_{\alpha\beta}=G_{\alpha\rho}u_\beta u^\rho+G_{\sigma\beta}u_\alpha u^\sigma+\tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}\epsilon^{\mu\sigma\lambda\rho}G_{\lambda\rho}u_\sigma u^\nu\ , \tag{4}$$

so that the two 4-vectors

$$G_\alpha=G_{\alpha\rho}u^\rho,\quad G^{*\mu}=\tfrac{1}{2}\epsilon^{\mu\sigma\lambda\rho}G_{\lambda\rho}u_\sigma\ , \tag{5}$$

satisfy the conditions

$$u_\alpha G^\alpha=u_\beta G^{*\beta}=0\ . \tag{5$'$}$$

We may write relation (4) as

$$G_{\alpha\beta}=G_\alpha u_\beta-G_\beta u_\alpha+\epsilon_{\alpha\beta\mu\nu}G^{*\mu}u^\nu\ . \tag{6}$$

Relation (6) has been known for a long time [3] but rarely used. It seems that people did not realize, with a noticeable exception [4], that expressions (5) and (6) constitute, in fact, an extension of G to a moving medium when u_μ is the 4-velocity of the medium (we assume that the velocity of light $c=1$ to satisfy the condition $u_\mu u^\mu=1$). In the rest frame where $u_\mu=\delta_{\mu0}$ relations (5) reduce to

$$G_j=G_{j0},\quad G^{*i}=\tfrac{1}{2}\epsilon^{0ijk}G_{jk}\ , \tag{7}$$

which are the non-null components of $G_{\alpha\beta}$.

II. COVARIANT ELECTROMAGNETISM IN MOVING MEDIA

A. Maxwell's equations and boundary conditions

Let $F_{\alpha\rho}$ and $G^{\alpha\beta}$ be the two antisymmetric field tensors with components

$$\begin{aligned}&E_{j0}=E_j,\quad F_{ij}=\epsilon_{0ijk}B^k\ ,\\&G^{j0}=D^j,\quad G^{ij}=\epsilon^{0ijk}H_k\ ,\end{aligned} \tag{8}$$

where E and H are the electric and magnetic fields, D is the electric displacement, and B is the magnetic induction.

In a stationary medium the Maxwell equations are

$$\partial_\alpha G^{\alpha\beta}=0,\quad \partial_\alpha F^{*\alpha\beta}=0\ , \tag{9}$$

where $F^{*\alpha\beta}$ is the dual tensor

$$F^{*\alpha\beta}=\tfrac{1}{2}\epsilon^{\alpha\beta\rho\nu}F_{\mu\nu}\ . \tag{10}$$

Then the technique described in the Introduction gives the Maxwell equations in a moving medium. First we consider the 4-vectors

$$\begin{aligned}&G^\alpha=G^{\alpha\beta}u_\beta,\quad G^{*\alpha}=\tfrac{1}{2}\epsilon^{\alpha\beta\mu\nu}u_\beta G_{\mu\nu}\ ,\\&F^{*\alpha}=F^{*\alpha\beta}u_\beta,\quad F^{**\alpha}=\tfrac{1}{2}\epsilon^{\alpha\beta\mu\nu}u_\beta F^*_{\mu\nu}\ ,\end{aligned} \tag{11}$$

and a simple calculation gives

$$F^{**\alpha}=F^\alpha\ . \tag{11$'$}$$

Then, using (6) and (11), Eqs. (9) become

$$\begin{aligned}&\partial_\alpha G^\alpha=0,\quad u^\alpha\partial_\alpha G^\beta=\epsilon^{\alpha\beta\mu\nu}u_\nu\partial_\alpha G^\alpha_\mu\ ,\\&\partial_\alpha F^{*\alpha}=0,\quad u^\alpha\partial_\alpha F^{*\beta}=\epsilon^{\alpha\beta\mu\nu}u_\nu\partial_\alpha F_\mu\ .\end{aligned} \tag{12}$$

The relations (12) are the Maxwell equations in a medium moving with the uniform velocity u_μ. Moreover, one checks easily that in the rest frame Eqs. (12) reduce to Eqs. (9).

Let us now consider the boundary conditions on a two-dimensional (2D) surface S with normal n_j. We may look at n_j as a 4-vector n_μ with the component n_0 null. As is well known, the boundary conditions in a stationary medium are

$$n_\alpha(G^{\alpha\beta}-\hat{G}^{\alpha\beta})=0, \quad n_\alpha(F^{*\alpha\beta}-\hat{F}^{*\alpha\beta})=0 , \tag{13}$$

where the pairs (G,F) and $(\hat{G},\hat{F})$ denote the electromagnetic field on opposite sides of the surface S. The comparison of Eqs. (9) with Eqs. (13) shows that one has only to change ∂_α into n_α in Eqs. (12) to obtain the boundary conditions in a moving medium. It becomes

$$\begin{aligned} &n_\alpha(G^\alpha-\hat{G}^\alpha)=0 , \\ &u^\alpha n_\alpha(G^\beta-\hat{G}^\beta)=\epsilon^{\alpha\beta\mu\nu}u_\nu n_\alpha(G^*_\mu-\hat{G}^*_\mu) , \\ &n_\alpha(F^\alpha-\hat{F}^\alpha)=0 , \\ &u^\alpha n_\alpha(F^{*\beta}-\hat{F}^{*\beta})=\epsilon^{\alpha\beta\mu\nu}u_\nu n_\alpha(F_\mu-\hat{F}_\mu) . \end{aligned} \tag{14}$$

These relations hold valid either for a fixed surface in a moving medium or for a surface moving uniformly in a stationary medium.

B. Constitutive relations

In a general linear medium with instantaneous and local interactions between the fields the constitutive relations take the form [1]

$$G^{\alpha\beta}=\tfrac{1}{2}\chi^{\alpha\beta,\lambda\nu}F_{\lambda\nu} , \tag{15}$$

where $\chi^{\alpha\beta,\lambda\nu}$ is a fourth-rank tensor satisfying the conditions

$$\begin{aligned} &\chi^{\alpha\beta,\lambda\nu}=-\chi^{\beta\alpha,\lambda\nu}, \quad \chi^{\alpha\beta,\lambda\nu}=-\chi^{\alpha\beta,\nu\lambda} , \\ &\chi^{\alpha\beta,\lambda\nu}=\chi^{\lambda\nu,\alpha\beta}, \quad \chi^{[\alpha\beta,\lambda\nu]}=0 , \end{aligned} \tag{16}$$

so that this tensor has only 20 independent components. Explicitly, one has

$$\chi^{0i,0j}=-\epsilon^{ij}, \quad \chi^{0i,jk}=\gamma^{il}, \quad \chi^{jkmn}=\kappa^{lp} , \tag{17}$$

where the triplets j,k,l and m,n,p are a circular permutation of 1,2,3. Moreover, we assume a nondispersive medium. Then the tensors ϵ and κ are real, but since γ is a pseudotensor it has to be pure imaginary.

Using (8) we get from (17) the constitutive relations for a stationary medium,

$$D^i=\epsilon^{ij}E_j+\gamma^{il}B_1, \quad H^i=\kappa^{ij}B_j+\gamma^{li}E_1 . \tag{18}$$

ϵ and κ are, respectively, the permittivity and the inverse of the permeability matrices, while γ and its transpose γ^T ($\gamma_{li}=\gamma^T_{il}$) are chirality matrices [5].

For a homogeneous medium moving with the uniform velocity u the constitutive relations are obtained with the technique already used in Sec. II A. From (5) and (15) we get $G^\alpha=\frac{1}{2}\chi^{\alpha\beta\lambda\nu}u_\rho F_{\lambda\nu}$ and using (6) for $F_{\mu\nu}$ it becomes

$$\begin{aligned} G^\alpha&=\tfrac{1}{2}\chi^{\alpha\beta,\lambda\nu}u_\beta(F_\lambda u_\nu-F_\nu u_\lambda+\tfrac{1}{2}\epsilon_{\lambda\mu\rho\sigma}F^{*\rho}u^\sigma) \\ &=\chi^{\alpha\beta,\lambda\nu}u_\beta u_\nu F_\lambda+\tfrac{1}{2}\epsilon_{\lambda\nu\rho\sigma}\chi^{\alpha\beta,\lambda\nu}u_\beta u^\sigma F^{*\rho}) . \end{aligned} \tag{19}$$

Then, with the tensors

$$\varphi^{\alpha\lambda}=\chi^{\alpha\beta,\lambda\nu}u_\beta u_\nu , \tag{20a}$$

$$\varphi^{*\alpha}_\rho=\tfrac{1}{2}\epsilon_{\lambda\nu\rho\sigma}\chi^{\alpha\beta,\lambda\nu}u_\beta u^\sigma , \tag{20b}$$

relation (19) becomes

$$G^\alpha=\varphi^{\alpha\lambda}F_\lambda+\varphi^{*\alpha}_\rho F^\rho . \tag{21}$$

In the same way one has, according to (11) and (15),

$$G^*_\alpha=\tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta}\chi^{\mu\nu\rho\sigma}u^\rho F_{\rho\sigma} , \tag{22}$$

and still using (6) for $F_{\rho\sigma}$,

$$\begin{aligned} G^*_\alpha&=\tfrac{1}{4}\epsilon_{\alpha\beta\mu\nu}\chi^{\mu\nu\rho\sigma}u^\beta(F_\rho u_\sigma-F_\sigma u_\rho+\epsilon_{\rho\sigma\lambda\tau}F^{*\lambda}u^\tau) \\ &=\tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}\chi^{\mu\nu\rho\sigma}u^\beta u_\sigma F_\rho \\ &\quad+\tfrac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon_{\rho\sigma\lambda\tau}\chi^{\mu\nu\rho\sigma}u^\beta u^\tau F^{*\lambda} . \end{aligned} \tag{22'}$$

Now with the tensor

$$\varphi^{**}_{\alpha\lambda}=\tfrac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon_{\rho\sigma\lambda\tau}\chi^{\mu\nu,\rho\sigma}u^\beta u^\tau , \tag{23}$$

and taking into account (20a), expression (22′) becomes

$$G^*_\alpha=\varphi^{*\rho}_\alpha F_\rho+\varphi^{**}_{\alpha\lambda}F^{*\lambda} . \tag{24}$$

Relations (21) and (24) are the constitutive relations of electromagnetism in a moving medium. In the rest frame these relations reduce to (18) since $u_\nu=\delta_{\nu_0}$ and since according to (5) and (8) one has

$$\begin{aligned} &G_j=G_{j0}=D_j, \quad G^*_j=\tfrac{1}{2}\epsilon_{0jkl}G^{kl}=H_j, \quad G_0=G^*_0=0 , \\ &F_j=F_{j0}=E_j, \quad F^*_j=\tfrac{1}{2}\epsilon_{0jkl}F^{kl}=G_j, \quad F_0=F^*_0=0 , \end{aligned} \tag{25}$$

while the non-null components of the tensors (20) and (23) are

$$\begin{aligned} &\varphi^{ij}=\chi^{i0,j0}=-\epsilon^{ij} , \\ &\varphi^{*i}_l=\tfrac{1}{2}\epsilon_{0jkl}\chi^{i0,jk}=-\gamma^i_l , \\ &\varphi^{**}_{il}=\tfrac{1}{4}\epsilon_{0ijk}\epsilon_{0mnl}\chi^{jk,mn}=\kappa_{il} . \end{aligned} \tag{26}$$

C. Constitutive equations in homogeneous isotropic media

In a homogeneous isotropic medium the non-null components of the tensor $\chi^{\alpha\beta,\mu\nu}$ are

$$\begin{aligned} &\chi^{i0,j0}=\epsilon g^{ij}, \quad \chi^{0i,jk}=\dot{\xi}\epsilon^{0ijk} , \\ &\chi^{ik,jl}=\kappa(g^{ik}g^{jl}-g^{il}g^{jk}) , \end{aligned} \tag{27}$$

with, in addition, those obtained by the symmetries (16). ϵ, κ, and $\dot{\xi}=i\xi$ are, respectively, the permittivity, the inverse of the permeability, and the chirality parameter.

Taking into account (27), the components of the tensors (20) and (23) may be easily calculated. With $u^2=-u_iu^i$ we get from (20a)

$$\varphi^{00}=-\epsilon u^2 ,$$
$$\varphi^{0i}=\varphi^{i0}=-\epsilon u^0 u^i , \tag{28a}$$
$$\varphi^{ij}=\epsilon u_0^2 g^{ij}-\kappa(u^2 g^{ij}+u^i u^j) ,$$

from (20b)

$$\varphi_0^{*0}=-\dot{\xi}u^2 ,$$
$$\varphi_0^{*i}=-\dot{\xi}u_0 u^i, \quad \varphi_i^{*0}=-\dot{\xi}u^0 u_i , \tag{28b}$$
$$\varphi_j^{*i}=\dot{\xi}(\rho_j^i-u^i u_j)+(\epsilon-\kappa)g^{ik}\epsilon_{0jkl}u_0 u^1 ,$$

and finally from (23)

$$\varphi_{00}^{**}=-\kappa u^2 ,$$
$$\varphi_{0i}^{**}=\varphi_{i0}^{**}=-\kappa u_0 u_i , \tag{28c}$$
$$\varphi_{ij}^{**}=\kappa u_0^2 g_{ij}-\epsilon(u^2 g_{ij}+u_i u_j) .$$

One may note the symmetry between these relations. One also checks that $u_\alpha \varphi^{\alpha\lambda}=u_\lambda \varphi^{\alpha\lambda}=0$ and similarly for the two other tensors.

Substituting (28) into (21) gives

$$G^0=\epsilon F^0+\dot{\xi}F^{*0} ,$$
$$G^i=(\kappa-\epsilon)(u_0 u^i F^0+\epsilon^{0ijk}u_0 u_k F_j^*) \tag{29}$$
$$+(\epsilon u_0^2-\kappa u^2)F^i+\dot{\xi}F^{*i} .$$

Similarly the relations (24) become

$$G_0=\kappa F_0^*+\dot{\xi}F_0 ,$$
$$G_j=(\epsilon-\kappa)(u_0 u_j F^{*0}+\epsilon_{0jkl}u^0 u^l F^k) \tag{30}$$
$$+(\kappa u_0^2-\epsilon u^2)F_j^*+\dot{\xi}F_j .$$

Then, using (8), (10), and (11), we obtain relations (29) and (30) in terms of the fields E, B, D, and H. We get from (29)

$$D^j u_j=\epsilon E^j u_j-\dot{\xi}B^j u_j , \tag{31a}$$
$$D^i u^0+\epsilon^{0ijk}u_j H_k=u^0(\epsilon E^i-\dot{\xi}B^i)$$
$$+\epsilon^{0ijk}u_j(\kappa B_k-\dot{\xi}E_k) \tag{31b}$$

and from (30)

$$H^j u_j=\kappa B^j u_j-\dot{\xi}E^j u_j , \tag{32a}$$
$$H_i u_0+\epsilon_{0ijk}u^j D^k=u_0(\kappa B_i-\dot{\xi}E_i)$$
$$+\epsilon_{0ijk}u^j(\epsilon E^k-\dot{\xi}B^k) . \tag{32b}$$

The relation (31a) [(32a)] is a consequence of (31b) [(32b)] and the relations (31) and (32) are invariant under the duality transformation

$$(D,B,\epsilon)\Longrightarrow(H,E,\kappa) . \tag{33}$$

For a stationary medium the relations (31b) and (32b) reduce to

$$D^i=\epsilon E^i-\dot{\xi}B^i , \tag{34}$$
$$H_i=\kappa B_i-\dot{\xi}E_i , \tag{35}$$

which have been used in many works [6] on electromagnetic wave propagation in isotropic chiral media.

III. ELECTROMAGNETIC CHIRAL FORMALISM

In the formalism of Sec. II the vectors E and D and the pseudovectors B and H are treated similarly, which is normal in a medium invariant under space inversions. But as already noticed when we discussed the constitutive relations, chirality breaks this invariance. So it is justifiable to look for a formalism treating vectors and pseudovectors differently. We now present such a formalism based on the two 4-vectors

$$P^\alpha=G^\alpha+i\eta F^{*\alpha}, \quad Q_\mu=G_\mu^*+iF_\mu . \tag{36}$$

In these relations $i=\sqrt{-1}$ and η is an arbitrary dimensional constant with the same dimension as ϵ, κ, and $\dot{\xi}$. As we shall see we may use this arbitrary constant to simplify the formalism and, for instance, for plane waves in a chiral isotropic medium a natural choice is $\eta=-\dot{\xi}$, which justifies the name given to this formalism.

With the 4-vectors (36), the Maxwell equations (12) become

$$\partial^\alpha P_\alpha=0, \quad u^\lambda\partial_\lambda P^\alpha=\epsilon^{\beta\alpha\mu\nu}\partial_\beta u_\nu Q_\mu , \tag{37}$$

and the solutions of Eqs. (37) may be obtained in terms of a Hertz 4-vector Π_μ by the relations

$$P^\alpha=\epsilon^{\beta\alpha\mu\nu}u_\nu\partial_\beta\Pi_\mu, \quad Q_\mu=u^\rho\partial_\rho\Pi_\mu , \tag{38}$$

so that if we define Π_μ as

$$\Pi_\mu=\Lambda_\mu+i\eta\Sigma_\mu , \tag{39}$$

we get from (36) and (38)

$$G^\alpha=\epsilon^{\lambda\alpha\mu\nu}u_\nu\partial_\lambda\Lambda_\mu, \quad G_\mu^*=u^\rho\partial_\rho\Lambda_\mu ,$$
$$F_\mu=u^\rho D_\rho\Sigma_\mu, \quad F^{*\alpha}=\epsilon^{\lambda\alpha\mu\nu}u_\nu\partial_\lambda\Sigma_\mu . \tag{40}$$

Then, substituting these expressions into the constitutive relations (21) and (24) gives

$$\epsilon^{\lambda\alpha\mu\nu}u_\nu\partial_\lambda\Lambda_\mu=\varphi^{\alpha\lambda}u^\rho\partial_\rho\Sigma_\lambda+\varphi_\rho^{*\alpha}\epsilon^{\lambda\rho\mu\nu}u_\nu\partial_\lambda\Sigma_\mu ,$$
$$u^\rho\partial_\rho\Lambda_\alpha=\varphi_\alpha^{*\rho}u^\sigma\partial_\sigma\Sigma_\rho+\varphi_{\alpha\lambda}^{**}\epsilon^{\rho\lambda\mu\nu}u_\nu\partial_\rho\Sigma_\mu . \tag{41}$$

Eliminating Λ_μ from these two relations leads to

$$(\varphi^{\alpha\mu}u^\sigma u^\rho\partial_\sigma\partial_\rho-\epsilon^{\sigma\alpha\tau\nu}\epsilon^{\rho\lambda\mu\beta}\varphi_{\tau\lambda}^{**}u_\beta u_\nu\partial_\rho\partial_\sigma)\Sigma_\mu-(\epsilon^{\sigma\alpha\lambda\nu}\varphi_\lambda^{*\mu}u_\nu u^\rho\partial_\rho\partial_\sigma+\epsilon^{\rho\lambda\mu\beta}\varphi_\lambda^{*\alpha}u_\beta u^\sigma\partial_\rho\partial_\sigma)\Sigma_\mu=0 , \tag{42}$$

which is the generalized wave equation satisfied by the η-part Σ_μ of the Hertz vector potential. Consequently, to obtain the solutions to Maxwell's equations one first has to solve the generalized wave equation (42) and second to calculate Λ_μ from (41). Then, knowing the Hertz potential, the relations (36) and (40) supply the electromagnetic field.

We also note that in terms of the 4-vectors P^α and Q_μ the boundary conditions (13) become

$$n_\alpha(P^\alpha-\hat{P}^\alpha)=0\ ,$$
$$u_\alpha n^\alpha(P^\beta-\hat{P}^\beta)=\epsilon^{\alpha\beta\mu\nu}u_\nu n_\alpha(Q_\mu-\hat{Q}_\mu)\ . \tag{43}$$

It has been known for a long time [7] that the Hertz potential is a powerful tool in chiral media, so the generalization discussed here is a natural one with the virtue to make calculations less cumbersome (see [4] for a comparison).

IV. PLANE WAVES IN A MOVING HOMOGENEOUS ISOTROPIC CHIRAL MEDIUM

A. Wave modes

We illustrate the chiral formalism on plane waves propagating in a homogeneous isotropic chiral medium moving with a uniform velocity. With all the components of the electromagnetic field proportional to $e^{ik_\mu x^\mu}$, all the results of the previous sections hold valid provided that one changes the derivative operator ∂_μ into iK_μ. Then the wave equation (42) becomes an algebraic equation:

$$(\varphi^{\alpha\mu}u^\rho u^\sigma K_\rho K_\sigma-\epsilon^{\sigma\alpha\tau\nu}\epsilon^{\rho\lambda\mu\beta}\varphi^{**}_{\tau\lambda}u_\beta u_\nu K_\rho K_\sigma)\Sigma_\mu -(\epsilon^{\sigma\alpha\lambda\nu}\varphi^{*\mu}_{\lambda}u^\rho K_\rho K_\sigma+\epsilon^{\rho\lambda\mu\beta}\varphi^{*\alpha}_{\lambda}u_\beta u^\sigma K_\rho K_\sigma)\Sigma_\mu=0\ , \tag{44}$$

and with the tensor

$$K^{\lambda\mu}=\epsilon^{\alpha\beta\lambda\mu}u_\alpha K_\beta\ , \tag{45}$$

Eq. (44) becomes

$$(u^\rho K_\rho\varphi^{\alpha\mu}-K^{\alpha\tau}\varphi^{**}_{\tau\lambda}K^{\lambda\mu})\Sigma_\mu +u^\rho K_\rho(K^{\alpha\lambda}\varphi^{*\mu}_{\lambda}+\varphi^{*\alpha}_{\lambda}K^{\lambda\mu})\Sigma_\mu=0\ . \tag{46}$$

We take the z axis in the direction of the velocity and we assume that the plane waves also propagate in this direction. Finally, the motion is assumed to be slow enough to make negligible the terms in β^2 ($\beta=v$ since $c=1$) so that one has

$$u_1=u_2=K_1=K_2=0,\quad u_3=-u^3=\beta,\quad u_0=u^0=1\ . \tag{47}$$

We also use the notations

$$K'_0=K^\rho u_\rho=K_0-\beta K_3,\quad K'_3=K_3-\beta K_0\ . \tag{47'}$$

Then, taking into account (47) and (47′), the components of the tensor $K^{\lambda\mu}$ become, according to (45),

$$K^{0\mu}=K^{\mu 0}=0,\quad K^{ij}=\epsilon^{03ij}K'_3\ , \tag{48}$$

while the components (28) of the tensors $\varphi^{\mu\nu}$, $\varphi^{*\nu}_{\mu}$, and $\varphi^{**}_{\mu\nu}$ reduce to

$$\varphi^{00}=0\ ,$$
$$\varphi^{0j}=\varphi^{j0}=-\epsilon\beta g^{j3}\ , \tag{49a}$$
$$\varphi^{ij}=\varphi^{ji}=\epsilon g^{ij}\ ,$$

$$\varphi^{**}_{00}=0\ ,$$
$$\varphi^{**}_{0j}=\varphi^{**}_{j0}=\kappa\beta g_{j3}\ , \tag{49b}$$
$$\varphi^{**}_{ij}=\varphi^{**}_{ji}=\kappa g_{ij}\ ,$$

and

$$\varphi^{*0}_{0}=0\ ,$$
$$\varphi^{*j}_{0}=-\varphi^{*0}_{j}=\xi\beta g^j_3\ , \tag{49c}$$
$$\varphi^{*i}_{j}=\xi g^i_j+\beta(\epsilon-\kappa)(g^{i1}g_{j2}-g^{i2}g_{j1})\ .$$

Then, using (48) and (49), the linear system (46) becomes

$$\beta\epsilon K_0^2\Sigma^3=0\ ,$$
$$(\epsilon K_0'^2-\kappa K_3'^2)\Sigma^1-2K'_0K'_3\xi\Sigma^2=0\ ,$$
$$(\epsilon K_0'^2-\kappa K_3'^2)\Sigma^2+2K'_0K'_3\xi\epsilon^1=0\ , \tag{50}$$
$$\epsilon K_0'^2\Sigma^3-\epsilon\beta K_0^2\Sigma^0=0\ .$$

The system (50) has nontrivial solutions Σ^1,Σ^2 with $\Sigma^0=\Sigma^3=0$ if the following dispersion relation is satisfied:

$$(\epsilon K_0'^2-\kappa K_3'^2)^2+4K_0'^2K_3'^2\xi^1=0\ , \tag{51}$$

that is, since $\xi=1\xi$, if

$$\epsilon K_0'^2-\kappa K_3'^2\pm 2\xi K'_0K'_3=0\ . \tag{51'}$$

Consequently, two different modes propagate in a moving chiral medium which are right and left circularly polarized waves. Using the definition (47′) of K'_0 and K'_3 the relation (51′) becomes

$$(\epsilon+2\beta\xi)K_0^2\pm 2K_0K_1(\xi+\beta\epsilon-\beta\kappa)-(\kappa+2\beta\xi)K_3^2=0\ ,$$

leading to

$$(\kappa+2\beta\xi)K_3=\pm K_0(\xi+\beta\epsilon-\beta\kappa\pm(\epsilon\kappa+\beta^2+4\beta\xi\epsilon)^{1/2})\ . \tag{52}$$

For $\xi=\beta=0$ one obtains the usual relation $K_3=n_0K_0$ with $n_0^2=\epsilon\kappa^{-1}$ and for $\beta=0$ one has

$$\kappa K_3=K_0(\pm\xi+(\epsilon\kappa+\xi^2)^{1/2})\ , \tag{53}$$

which is a well-known result [8]. In the same way we get from (52) for $\xi=0$

$$n_\pm=n_0\pm\beta(n_0^2-1)\ , \tag{54}$$

and more generally we deduce from (52) the refractive index

$$n_\pm=[n_0^2+\kappa^{-2}(\xi^2+4\beta\xi\epsilon)]^{1/2} \pm[\beta\kappa^{-1}+\beta(n^2-1-2\xi\kappa^{-1})]\ , \tag{55}$$

which is not in agreement with a previous result [4].

B. Electromagnetic fields

Assuming the condition (51′) is fulfilled, the solutions (50) are

$$\Sigma^0=\Sigma^3=0,\quad \Sigma^{2\pm}=\pm i\Sigma^1\ ; \tag{56}$$

according to (52) and (55)

$$K_3=K_3^+=n^+K_0\quad\text{for }\Sigma^2=i\Sigma^1\ ,$$
$$K_3=K_3^-=n^-K_0\quad\text{for }\Sigma^2=-i\Sigma^1\ . \tag{56'}$$

Now for the plane waves the second equation (41) becomes

$$K_0'\Lambda_\alpha=K_0'\varphi_\alpha^{*\rho}\Sigma_\rho+\varphi_{\alpha\mu}^{**}\epsilon^{\rho\lambda\mu\nu}u_\nu K_\rho\Sigma_\mu \ . \tag{57}$$

Substituting (56) into (57) and using (49c), we get

$$\begin{aligned}&\Lambda_0=\Lambda_3=0 \ ,\\ &\Lambda_1^\pm=\dot{\xi}\Sigma_1\pm i\kappa\left[\frac{K_3'}{K_0'}-\beta\left[\frac{K_3^2}{K_0^2}-1\right]\right]\Sigma_1 \ ,\\ &\Lambda_2^\pm=\pm i\dot{\xi}\Sigma_1-\kappa\left[\frac{K_3'}{K_0'}-\beta\left[\frac{K_3^2}{K_0^2}-1\right]\right]\Sigma_1 \ ,\end{aligned} \tag{58}$$

and using (47′) a simple calculation gives

$$\frac{K_3'}{K_0'}-\beta\left[\frac{K_3^2}{K_0^2}-1\right]=\frac{K_3}{K_0}+O(\beta^2) \ , \tag{58'}$$

where $O(\beta^2)$ means that one neglects the terms of order superior to β. Then, taking into account (56′) one has

$$\begin{aligned}&\Lambda_0=\Lambda_3=0 \ ,\\ &\Lambda_1^\pm=\dot{\xi}\Sigma_1+iz^\pm\Sigma_1 \ ,\\ &\Lambda_2^\pm=\pm i\dot{\xi}-z^\pm\Sigma_1 \ ,\end{aligned} \tag{59}$$

where $z^\pm=\kappa n^\pm$ is the wave impedance.

Now, using (56) and (59), the components of the Hertz vector potential are, according to (39),

$$\begin{aligned}&\Pi_0=\Pi_3=0 \ ,\\ &\Pi_1^\pm=i(\xi+\eta\pm z^\pm)\Sigma_1 \ ,\\ &\Pi_2^\pm=[\mp(\xi+\eta)-z^\pm]\Sigma_1 \ .\end{aligned} \tag{60}$$

Since the parameter η is arbitrary, a natural choice to simplify (60) is $\eta=-\xi$ and the expressions (60) reduce to

$$\begin{aligned}&\Pi_0=\Pi_3=0 \ ,\\ &\Pi_1^\pm=\pm iz^\pm\Sigma_1 \ ,\\ &\Pi_2^\pm=-z^\pm\Sigma_1 \ .\end{aligned} \tag{61}$$

Then, substituting (61) into (38) in which ∂_λ has been replaced by K_λ gives the components of the electromagnetic field:

$$\begin{aligned}&Q_0=Q_3=0,\quad Q_1^\pm=\pm iK_0'z^\pm\Sigma_1,\quad Q_2^\pm=-K_0'z^\pm\Sigma_1 \ ,\\ &P^0=P^3=0,\quad P^{1,\pm}=-K_3'z^\pm\Sigma_1,\quad P^{2,\pm}=\mp iK_3'z^\pm\Sigma_1 \ .\end{aligned} \tag{62}$$

As already noted, these solutions correspond to right and left circularly polarized plane waves.

C. Mode excitation

A linearly polarized plane wave incident normally on the boundary S of a moving medium comes from a homogeneous achiral medium with refractive index $\hat{n}^2=\hat{\epsilon}\hat{\kappa}^{-1}$. In a moving medium the electromagnetic field is characterized by the phase $\exp(iK_0x_0-iK_3^\pm x_3)$, with $K_3^\pm=n^\pm K_0$, $n^\pm$ given by (55) and by the amplitudes (62) $Q_\mu^\pm$ and $P^{\alpha,\pm}$, in which Σ_1 is an arbitrary constant.

The phase of the incident plane wave is $\exp(K_0x_0-\hat{K}_3x_3)$, with $\hat{K}_3=\hat{n}K_0=(\hat{\epsilon}\hat{\kappa}^{-1})^{1/2}K_0$, and the equality of phases on both sides of the boundary S implies

$$K_3^\pm=n^\pm(\hat{n})^{-1}\hat{K}_3 \ . \tag{63}$$

This relation determines the wave numbers $K_3^\pm$ in the moving medium in terms of the incident wave number $\hat{K}_3$.

Of course the components of the incident electromagnetic field may be obtained by using the chiral formalism. In fact a solution of the system (50) is

$$\hat{\Sigma}^0=\hat{\Sigma}^2=\hat{\Sigma}^3=0,\quad \hat{\Sigma}^1 \text{ arbitrary} \ . \tag{64}$$

This leads to

$$\hat{\Lambda}_0=\hat{\Lambda}_1=\hat{\Lambda}_3=0,\quad \hat{\Lambda}_2=-\hat{z}\hat{\Sigma}_1,\quad \hat{z}=(\hat{\epsilon}\hat{\kappa})^{1/2} \ . \tag{65}$$

The components of the Hertz vector potential are

$$\hat{\Pi}_0=\hat{\Pi}_3=0,\quad \hat{\Pi}_1=i\hat{\eta}\hat{\Sigma}_1,\quad \hat{\Pi}_2=-\hat{z}\hat{\Sigma}_1 \ . \tag{66}$$

Since $\hat{\eta}$ is arbitrary the comparison of $\hat{\Pi}_1$ and $\hat{\Pi}_2$ suggests taking $\hat{\eta}=\pm\hat{z}$. So in the chiral formalism a linearly polarized plane wave is divided in a natural way into two circularly polarized plane waves with equal amplitude and wave number. Amplitudes of the incident electromagnetic field are

$$\begin{aligned}&\hat{Q}_0=\hat{Q}_3=0,\quad Q_1^\pm=\pm iK_0\hat{z}\hat{\Sigma}_1,\quad \hat{Q}_2^\pm=-K_0\hat{z}\hat{\Sigma}_1 \ ,\\ &\hat{P}^0=\hat{P}^3=0,\quad p^{1\pm}=-\hat{K}_3\hat{z}\hat{\Sigma}_1,\quad p^{2,\pm}=\mp i\hat{K}_3\hat{z}\hat{\Sigma}_1 \ .\end{aligned} \tag{67}$$

Let us now consider the boundary conditions (43). Since the components of the normal 4-vector are (0,0,1,0), we get from (43)

$$n_3(P^3-\hat{P}^3)=0 \ , \tag{68a}$$

$$\beta(P^1-\hat{P}^1)=-(Q_2-\hat{Q}_2) \ , \tag{68b}$$

$$\beta(P^2-\hat{P}^2)=Q_1-\hat{Q}_1 \ . \tag{68c}$$

From (62) and (67) one sees at once that (68a) is identically satisfied while (68b) and (68c) supply the same condition,

$$K_0Z^\pm\Sigma_1=(K_0+\beta\hat{K}_3)\hat{Z}^\pm\hat{\Sigma}_1 \ , \tag{69}$$

provided, or course, that chirality on both sides of S is correctly matched. The relation (69) defines the amplitudes of the electromagnetic field in the moving medium in terms of the incident electromagnetic field.

V. CONCLUSION

The manifestly covariant formalism discussed in Sec. II presents the advantage of being in a very compact form (making sure that the equations and boundary equations to be used are the correct ones), which is an important result. But this compact form is not well suited to the solutions of practical problems. That is why we developed the chiral formalism, supplying a generalized wave equation which leads with the boundary conditions to a well-posed problem in Hadamard's sense. One may rightly argue that it is difficult to solve such an equation. But on one hand the wave equation (42) simplifies considerably in practical problems since anisotropy intervenes only in permittivity, as in chiroplasmas, or in permeability, as in chiroferrites [9]. On the other hand there exists a great diversity of mathematical and numerical works on this type of equation.

We were able to solve completely the problem of plane waves propagating in a moving homogeneous isotropic medium provided that one assumes the motion slow enough with respect to the light velocity and provided that the plane waves propagate in the same direction as the medium. From a practical point of view the first assumption is not very restrictive. By contrast, the second one is essential to obtain the exact modes of propagation. Otherwise the dispersion relation is the solution of an equation of sixth degree.

To conclude, one may state that even if approximate methods are required to solve a problem in the chiral formalism one shall have only to check the consistency of the approximations, since the basic equations are correct. Of course, for a stationary medium, the present formalism reduces to the formalisms previously discussed in [10] and [11] for isotropic and anisotropic media, respectively.

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